

SOLUTION EXERCISE SHEET 10

Exercise 1. Since U is open, we know that for any $z \in K$ there exists an $\varepsilon_z > 0$ such that $D_{\varepsilon_z}(z) \subset U$. Set $\delta_z = \frac{1}{2}\varepsilon_z$. Then, given that $D_{\delta_z}(z)$ is an open cover of the compact set K , there exists a finite subcover $\{D_{\delta_{z_j}}(z_j)\}_{j=1,\dots,N}$ such that

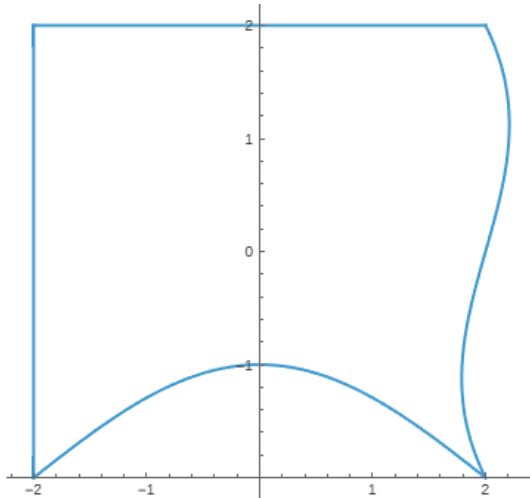
$$K \subset \bigcup_{j=1}^N D_{\delta_{z_j}}(z_j) =: V.$$

Then, by construction $\bar{V} \subset \bigcup_{j=1}^N D_{2\delta_{z_j}}(z_j) \subset U$. Note that \bar{V} is also compact. This follows directly from the Heine-Borel theorem. By employing the same arguments as before, we can find a cover of \bar{V} by finitely many balls $\{D_{\frac{1}{2}r_j}(\xi_j)\}_{j=1,\dots,M}$, where the r_j are chosen such that

$$D_{r_j}(\xi_j) \subset U.$$

Set, $\delta = \frac{1}{2} \min\{r_1, r_2, \dots, r_M\}$. Then, for any $q \in \bar{V}$ there exists a $j \in \{1, \dots, M\}$ such that $q \in D_{\frac{1}{2}r_j}(\xi_j)$. From the triangle inequality it then follows that $D(q, \delta) \subset U$.

Exercise 2. Note that the path in question looks as follows.



In particular, one readily computes that γ is a piecewise smooth closed path. First, we define

$$\varphi : [0, 1] \rightarrow [0, 4], \quad \varphi(t) = 4t$$

and set

$$\gamma_0(t) = \gamma \circ \varphi(t)$$

and recall that

$$\oint_{\gamma_0} f(z)dz = \oint_{\gamma} f(z)dz.$$

Next, we define the homotopy

$$\Gamma(s, t) = (1 - s)\gamma_0(t) + se^{2\pi it + \pi i}.$$

Then, since $f(z) = \frac{\cos(z)}{z} - \frac{\sin(z)}{z^4}$ is holomorphic on \mathbb{C}_* and, as $\Gamma(s, t) \neq 0$ for all $(s, t) \in [0, 1]^2$, we conclude that

$$\oint_{\gamma} f(z) = \oint_{\gamma_1} f(z)$$

where $\gamma_1(t) = \Gamma(1, t) = e^{2\pi it + \pi i}$. Now, one way of computing the integral is to invoke the Cauchy integral theorem to compute that

$$\begin{aligned} \oint_{\gamma_1} \frac{\cos(z)}{z} dz &= 2\pi i \cos(0) = 2\pi i, \\ - \oint_{\gamma_1} \frac{\sin(z)}{z^4} dz &= -\frac{2\pi i}{3!} \frac{\partial^3 \sin(z)}{\partial z^3} \Big|_{z=0} = \frac{\pi i}{3} \cos(0) = \frac{\pi i}{3}, \end{aligned}$$

and conclude that

$$\oint_{\gamma} f(z) dz = \frac{7\pi i}{3}.$$

Alternatively, one performs a Taylor expansion and computes the integral term by term. The result is then a straightforward consequence of the fact

$$\oint_{\gamma_1} z^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{-1\}. \end{cases}$$

Exercise 3. We start by rewriting f as

$$f(z) = \frac{1}{z^2 - (3+i)z + 2 + 2i} = \frac{1}{(z-2)(z-(1+i))}.$$

Then, as neither pole lies on the contour that we are concerned with, the integral is well defined. Next, by doing a partial fraction decomposition, we see that

$$f(z) = \frac{1}{(z-2)(z-(1+i))} = \frac{1}{1-i} \left(\frac{1}{z-2} - \frac{1}{z-(1+i)} \right).$$

Hence, f is holomorphic up to two points. Note that a parameterization of the ellipse in question is given by

$$\gamma(t) = \sqrt{10} \cos(t) + i\sqrt{20} \sin(t).$$

Therefore, we can use the homotopy

$$\Gamma(s, t) = \sqrt{10} \cos(t) + i\sqrt{10(1+s)} \sin(t)$$

to obtain an integral around the circle of radius $\sqrt{10}$, i.e., along the curve

$$\gamma_0(t) = \sqrt{10} e^{2\pi it}.$$

Now, there are several ways to see that the integral in question is equal to 0. The fastest way is to apply the Cauchy integral theorem to the constant function $f(z) = 1$.

Another, less sophisticated way is as follows. We define two more homotopies, namely

$$\Gamma_1(s, t) := \sqrt{10}e^{2\pi it} + 2s.$$

$$\Gamma_2(s, t) := \sqrt{10}e^{2\pi it} - s(1 + i)$$

These homotopies translate the circle. In particular, one computes that

$$\frac{1}{1-i} \oint_{\Gamma(0,t)} \frac{1}{z-2} dz = \frac{1}{1-i} \oint_{\Gamma_1(1,t)} \frac{1}{z-2} dz = \frac{2\pi i}{1-i} \int_0^1 dz = \frac{2\pi i}{1-i}$$

and

$$-\frac{1}{1-i} \oint_{\Gamma(0,t)} \frac{1}{z-(1+i)} dz = -\frac{1}{1-i} \oint_{\Gamma_2(1,t)} \frac{1}{z-(1+i)} dz = -\frac{2\pi i}{1-i} \int_0^1 dz = -\frac{2\pi i}{1-i}.$$